VARIATIONAL METHOD OF CALCULATING A NONSTATIONARY TEMPERATURE FIELD

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Variational formulation is given of the nonstationary heat conduction problem for two bodies; its numerical solution by the finite-element method (FEM) reduces to the solving of simultaneous linear algebraic equations.

1. Formulation of the Problem. A system of two bodies is given which are in contact with one another, each occupying the space V_1 and V_2 respectively with the corresponding boundaries B_1 and B_2. Let S be the common part of the boundary $B_1 + B_2$ (the surface of the contact) where one has a boundary condition of the third kind, Γ_1 and Γ_2 being the free parts of the boundary for which the heat flux is zero.

One then has the following system of equations:

$$c_1\gamma_1\dot{\theta}_1 = \lambda_1\theta_{1,ii} + \gamma_1 w_1 \quad (t > 0,\ \mathbf{x} \in V_1), \tag{1a}$$

$$c_2\gamma_2\dot{\theta}_2 = \lambda_2\theta_{2,ii} \quad (t > 0,\ \mathbf{x} \in V_2) \tag{1b}$$

with the initial conditions

$$\theta_1(\mathbf{x},\ 0) = \theta_1^0(\mathbf{x}),\ \theta_2(\mathbf{x},\ 0) = \theta_2^0(\mathbf{x}) \tag{2}$$

and the boundary conditions

$$\lambda_1\theta_{1,i}n_i^{(1)} = -\alpha(\theta_1 - \theta_2) \text{ on } S, \tag{3a}$$

$$\lambda_2\theta_{2,i}n_i^{(2)} = -\alpha(\theta_2 - \theta_1) \text{ on } S, \tag{3b}$$

$$\lambda_1\theta_{1,i}n_i^{(1)} = 0 \text{ on } \Gamma_1, \tag{4a}$$

$$\lambda_2\theta_{2,i}n_i^{(2)} = 0 \text{ on } \Gamma_2, \tag{4b}$$

where $B_1 = \Gamma_1 + S$, $B_2 = \Gamma_2 + S$. In the physical interpretation the subscript, or superscript, "1" corresponds to the metal, and the subscript "2" to the rollers.

2. Variational Formulation. Following [5] if the concept of convolution of two continuous functions $f(\mathbf{x}, t)$ and $g(\mathbf{x}, t)$ defined on $V \times [0, \infty]$ is introduced by means of

$$[f * g](\mathbf{x},\ t) = \int_0^t f(\mathbf{x},\ t - \tau)\, g(\mathbf{x},\ \tau)\, d\tau,\ (\mathbf{x},\ \tau) \in V \times [0,\ \infty), \tag{5}$$

where $V \times [0, \infty)$ denotes the set which is the direct product of the region V and of the time interval $[0, \infty)$ it can be seen that Eqs. (1) with the initial condition (2) are equivalent to the following relations:

$$c_1\gamma_1\theta_1 = \lambda_1 * \theta_{1,ii} + c_1\gamma_1\theta_1^0 + \gamma_1 * w_1 \text{ on } V_1 \times [0,\ \infty), \tag{6a}$$

$$c_2\gamma_2\theta_2 = \lambda_2 * \theta_{2,ii} + c_2\gamma_2\theta_2^0 \text{ on } V_2 \times [0,\ \infty). \tag{6b}$$

Indeed, by integrating, say, both sides of Eq. (1a) over $[0, t]$ and using (5) one obtains

$$c_1\gamma_1 \int_0^t \dot{\theta}_1(\mathbf{x},\ \tau)\, d\tau = \int_0^t (\lambda_1\theta_{1,ii} + \gamma_1 w_1)\, d\tau$$

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or

$$c_1\gamma_1\theta_1(\mathbf{x},\ t) - c_1\gamma_1\theta_1(\mathbf{x},\ 0) = \lambda_1 * \theta_{1,ii} + \gamma_1 * w_1.$$

The boundary conditions (3) and (4) can also be transformed in a similar manner resulting in equivalent relations:

$$\lambda_1 * \theta_{1,i} n_i^{(1)} = -\alpha * (\theta_1 - \theta_2) \text{ on } S \times [0,\ \infty), \tag{7a}$$

$$\lambda_2 * \theta_{2,i} n_i^{(2)} = -\alpha * (\theta_2 - \theta_1) \text{ on } S \times [0,\ \infty), \tag{7b}$$

$$\lambda_1 * \theta_{1,i} n_i^{(1)} = 0 \text{ on } \Gamma_1 \times [0,\ \infty), \tag{8a}$$

$$\lambda_2 * \theta_{2,i} n_i^{(2)} = 0 \text{ on } \Gamma_2 \times [0,\ \infty). \tag{8b}$$

To obtain the solution of the variational problem the concept of a feasible state $R = \{\theta_1, \theta_2\}$ is introduced which represents the totality of two continuous functions $\theta_1(\mathbf{x}, t)$ and $\theta_2(\mathbf{x}, t)$ defined on $V_1 \times [0, \infty)$ and $V_2 \times [0, \infty)$. Then the solution of the problem (1)-(4) can be determined as a feasible state $R = \{\theta_1, \theta_2\}$ such that it satisfies Eq. (1), the initial conditions (2) and the boundary conditions (3) and (4).

Further, let there be defined on the set of feasible states K for any $t \in [0, \infty)$ the functional

$$\begin{aligned}\Phi_t\{R\} = \frac{1}{2}\int_{V_1} [c_1\gamma_1\theta_1 * \theta_1 + \lambda_1 * \theta_{1,i} * \theta_{1,i} - 2c_1\gamma_1\theta_1^0 * \theta_1 - 2\gamma_1 * w_1 * \theta_1](\mathbf{x},\ t)\,dV_1 \\ + \frac{1}{2}\int_{V_2} [c_2\gamma_2\theta_2 * \theta_2 + \lambda_2 * \theta_{2,i} * \theta_{2,i} - 2c_2\gamma_2\theta_2^0 * \theta_2](\mathbf{x},\ t)\,dV_2 \\ + \frac{1}{2}\int_S [\alpha * (\theta_1 - \theta_2) * \theta_1](\mathbf{x},\ t)\,dS + \frac{1}{2}\int_S [\alpha * (\theta_2 - \theta_1) * \theta_2](\mathbf{x},\ t)\,dS.\end{aligned} \tag{9}$$

Variational formulation is now given which is a generalization of the principles originally formulated for problems of linear elastodynamics [2, 3].

<u>THEOREM.</u> For a feasible state $R = \{\theta_1, \theta_2\}$, $R \in K$ to be a solution of the problem (1)-(4) it is necessary and sufficient that on K the condition be satisfied

$$\delta\Phi_t\{R\} = 0 \quad (0 \leqslant t < \infty). \tag{10}$$

<u>Proof.</u> Having determined the variation of the functional (9) by taking into account the properties of the convolution [5] and the Gauss-Ostrogradskii theorem one obtains

$$\begin{aligned}\delta\Phi_t\{R\} = \int_{V_1} [(c_1\gamma_1\theta_1 - \lambda_1 * \theta_{1,ii} - c_1\gamma_1\theta_1^0 - \gamma_1 * w_1) * \delta\theta_1](\mathbf{x},\ t)\,dV_1 \\ + \int_{V_2} [(c_2\gamma_2\theta_2 - \lambda_2 * \theta_{2,ii} - c_2\gamma_2\theta_2^0) * \delta\theta_2](\mathbf{x},\ t)\,dV_2 + \int_{\Gamma_1} [\lambda_1 * \theta_{1,i} n_i^{(1)} * \delta\theta_1](\mathbf{x},\ t)\,d\Gamma_1 \\ + \int_S \{[\lambda_1 * \theta_{1,i} n_i^{(1)} + \alpha * (\theta_1 - \theta_2)] * \delta\theta_1\}(\mathbf{x},\ t)\,dS + \int_{\Gamma_2} [\lambda_2 * \theta_{2,i} n_i^{(2)} * \delta\theta_2](\mathbf{x},\ t)\,d\Gamma_2 + \int_S \{[\lambda_2 * \theta_{2,i} n_i^{(2)} + \alpha * (\theta_2 - \theta_1)] * \delta\theta_2\}(\mathbf{x},\ t)\,dS.\end{aligned} \tag{11}$$

It is obvious that the variational formulation is necessary since by assuming $R = \{\theta_1, \theta_2\}$ to be the solution of (1)-(4) together with (6)-(8) it can easily be seen that (10) follows from (11).

To prove the sufficiency two lemmas are formulated [3] similar to the fundamental lemma of the calculus of variations.

<u>LEMMA 1.</u> Let f be a continuous function $V \times [0, \infty)$ and let us assume that

$$\int_V [f * g](\mathbf{x},\ t)\,dV = 0 \quad (0 \leqslant t < \infty)$$

for any function g which vanishes on $B \times [0, \infty)$ (B is the boundary of the region V). Then $f = 0$ on $V \times [0, \infty)$.

<u>LEMMA 2.</u> Let f be a piecewise-continuous function on $B_1 \times [0, \infty)$ and let us assume that

$$\int_{B_1} [f * g](\mathbf{x},\ t)\,dB_1 = 0 \quad (0 \leqslant t < \infty).$$

for any function g which vanishes on $B_2 \times [0, \infty)$ (B_1 and B_2 are closed nonoverlapping subsets of the boundary B). Then f = 0 on $B_1 \times [0, \infty)$.

Let us assume the opposite, that is, that $R \in K$ satisfies (10). Then by choosing $\delta R = \{\delta\theta_1, 0\}$ such that $\delta\theta_1$ vanishes on $B_1 \times [0, \infty)$ it follows from (10) and (11) that

$$\int_{V_1} [(c_1\gamma_1\theta_1 - \lambda_1 * \theta_{1,ii} - c_1\gamma_1\theta_1^0 - \gamma_1 * w_1) * \delta\theta_1](x, t)\, dV_1 = 0, \tag{12}$$

and hence with the aid of Lemma 1 one obtains Eq. (6a).

Then choosing $\delta R = \{\delta\theta_1, 0\}$ such that $\delta\theta_1$ vanishes on $\Gamma_1 \times [0, \infty)$ one obtains from (10), (11), and (6a) with the aid of Lemma 2

$$\lambda_1 * \theta_{1,i} n_i^{(1)} = -\alpha * (\theta_1 - \theta_2) \text{ on } S \times [0, \infty). \tag{13}$$

For $\delta R = \{\delta\theta_1, 0\}$ such that $\delta\theta_1$ vanishes on $S \times [0, \infty)$ it follows from (10), (11), and (6a) that

$$\lambda_1 * \theta_{1,i} n_i^{(1)} = 0 \text{ on } \Gamma_1 \times [0, \infty). \tag{14}$$

Similarly, by choosing $\delta R = \{0, \delta\theta_2\}$ one can obtain Eq. (6b) from (10) and (11) as well as the corresponding boundary conditions (7b) and (8b).

Thus $R = \{\theta_1, \theta_2\}$ which satisfies the condition (10) is a solution of the problem (1)-(4).

3. The Finite-Element Method. The progress which has been made in solving elasticity theory problems by using the FEM [1] gives us reason to believe that the method will also prove fruitful for solving the heat-conduction problems.

The main idea of FEM consists in replacing the sought continuous space function by a finite number of its values defined at nodes of a grid. To this end the region of the continuum under consideration is subdivided into a number of elements (subregions) which are joined together on their boundaries in a finite number of points. The temperature field is approximated for each element by an algebraic polynomial which specifies in a unique manner the temperature in the interior of an element by the temperatures of the nodes corresponding to it.

The actual form of the approximation of the temperature field inside an element depends on the type of the elements used; therefore, in applications one usually selects simple geometric patterns such as a portion of a straight line for a one-dimensional problem, or a triangle for a planar, a tetrahedron for a spatial problem, etc.

The temperature at any point inside the m-th element can generally be expressed in terms of the temperatures of the nodes corresponding to it by means of the following matrix equation:

$$\theta_m(\mathbf{x}, t) = \langle b_m(\mathbf{x}) \rangle \{\theta(t)\}, \tag{15}$$

where $\{\theta(t)\}$ is the column-vector of node temperatures for the entire system of finite elements (the dimension of the vector is given by the full number of nodes for the entire discretized region); $\langle b_m(\mathbf{x}) \rangle$ is the row-vector of the form function (of the spatial approximation) which relates the current point coordinate inside the elements to the coordinates of the node.

It should be mentioned that the majority of the components of the vector $\langle b_m(\mathbf{x}) \rangle$ vanishes since the temperature within an element is usually determined by the temperature of the adjoining nodes.

Differentiating (15) with respect to the space coordinates one obtains the gradient vector of the temperature,

$$\{\theta_{m,i}(\mathbf{x}, t)\} = [a_m(\mathbf{x})]\{\theta(t)\}, \tag{16}$$

where the rectangular matrix $[a_m(\mathbf{x})]$ has been obtained by differentiating the row-vector $\langle b_m(\mathbf{x}) \rangle$.

The adopted temperature function should automatically ensure the continuity of the temperature field at the nodes and on the boundaries with adjacent elements.

For the boundary elements on the contact surface the temperature field is approximated by means of another form function which corresponds now to the surface distribution of temperature. Thus instead of (15) one sets

$$\theta_m(x, t) = \langle d_m(x) \rangle \{\theta(t)\}, \tag{17}$$

where the components of the row-vector $\langle d_m(x) \rangle$ are determined similarly as $\langle b_m(x) \rangle$.

By substituting (15)-(17) into the functional (9) one obtains

$$\Phi_t\{\theta_1, \theta_2\} = \sum_{m=1}^{M_1} \frac{1}{2} \Big\{ \Big\{ \int_{V_{1m}} [c_1\gamma_1 \{\theta_1(t)\}^T \langle b_{1m}(\mathbf{x}) \rangle^T * \langle b_{1m}(\mathbf{x}) \rangle \{\theta_1(t)\}$$

$$+ \lambda_1 * \{\theta_1(t)\}^T [a_{1m}(\mathbf{x})]^T * [a_{1m}(\mathbf{x})] \{\theta_1(t)\} - 2c_1\gamma_1 \{\theta_1(t)\}^T \langle b_{1m}(\mathbf{x}) \rangle^T * \langle b_{1m}(\mathbf{x}) \rangle$$

$$\times \{\theta_1^0\} - 2\gamma_1 * w_1 * \{\theta_1(t)\}^T \langle b_{1m}(\mathbf{x}) \rangle^T] \, dV_{1m} + \int_{S_m} [\alpha * (\langle d_{1m}(x) \rangle \{\theta_1(t)\}$$

$$- \langle d_{2m}(x) \rangle \{\theta_2(t)\}) * \{\theta_1(t)\}^T \langle d_{1m}(x) \rangle^T] \, dS_m \Big\} \Big\} + \sum_{m=1}^{M_2} \frac{1}{2} \Big\{ \Big\{ \int_{V_{2m}} [c_2\gamma_2 \{\theta_2(t)\}^T \langle b_{2m}(\mathbf{x}) \rangle^T * \langle b_{2m}(\mathbf{x}) \rangle \{\theta_2(t)\}$$

$$+ \lambda_2 * \{\theta_2(t)\}^T [a_{2m}(\mathbf{x})]^T * [a_{2m}(\mathbf{x})] \{\theta_2(t)\} - 2c_2\gamma_2 \{\theta_2(t)\}^T \langle b_{2m}(\mathbf{x}) \rangle^T$$

$$* \langle b_{2m}(\mathbf{x}) \rangle \{\theta_2^0\}] \, dV_{2m} + \int_{S_m} [\alpha * (\langle d_{2m}(x) \rangle \{\theta_2(t)\}$$

$$- \langle d_{1m}(x) \rangle \{\theta_1(t)\}) * \{\theta_2(t)\}^T \langle d_{2m}(x) \rangle^T] \, dS_m \Big\} \Big\} = \frac{1}{2} \{\theta_1(t)\}^T * [C_1] \{\theta_1(t)\}$$

$$+ \frac{1}{2} \{\theta_1(t)\}^T * [K_1] * \{\theta_1(t)\} - \{\theta_1(t)\}^T * [C_1] \{\theta_1^0\} - \{\theta_1(t)\}^T * \{Q_1(t)\}$$

$$- \frac{1}{2} \{\theta_1(t)\}^T * [L_{12}] * \{\theta_2(t)\} + \frac{1}{2} \{\theta_2(t)\}^T * [C_2] \{\theta_2(t)\}$$

$$+ \frac{1}{2} \{\theta_2(t)\}^T * [K_2] * \{\theta_2(t)\} - \{\theta_2(t)\}^T * [C_2] \{\theta_2^0\} - \frac{1}{2} \{\theta_2(t)\}^T * [L_{21}] * \{\theta_1(t)\} \quad (18)$$

(T is the transposition symbol) where M_1 and M_2 show the number of elements into which the first and the second body has been subdivided;

$$[C_n] = \sum_{m=1}^{M_n} [C_n^m] \quad (19a)$$

is the matrix of the heat capacity of the n-th body (n = 1, 2);

$$[K_n] = \sum_{m=1}^{M_n} [K_n^m] \quad (19b)$$

is the matrix of the heat conductivity of the n-th body;

$$[L_{np}] = \sum_{m=1}^{M_n} [L_{np}^m] \quad (19c)$$

is the matrix of the heat emission between the n-th and the p-th body (p = 2, 1);

$$\{Q_1(t)\} = \sum_{m=1}^{M_1} \{Q_1^m(t)\} \quad (19d)$$

is the vector of the thermal force of the first body.

The corresponding submatrices for the m-th element are determined in the following manner:

$$[C_n^m] = \int_{V_{nm}} c_n\gamma_n \langle b_{nm}(\mathbf{x}) \rangle^T \langle b_{nm}(\mathbf{x}) \rangle \, dV_{nm}, \quad (20a)$$

$$[K_n^m] = \int_{V_{nm}} \lambda_n [a_{nm}(\mathbf{x})]^T [a_{nm}(\mathbf{x})] dV_{nm} + \int_{S_{nm}} \alpha \langle d_{nm}(x) \rangle^T \langle d_{nm}(x) \rangle \, dS_{nm}, \quad (20b)$$

$$[L_{np}^m] = \int_{S_{nm}} \alpha \langle d_{nm}(x) \rangle^T \langle d_{pm}(x) \rangle \, dS_{nm}, \quad (20c)$$

$$\{Q_1^m(t)\} = \int_{V_{1m}} \gamma_1 * w_1 \langle b_{1m}(\mathbf{x}) \rangle^T \, dV_{1m}. \quad (20d)$$

The surface integral in (20b) only appears for the boundary elements adjoining the contact surface. In [4] an explicit expression was given for the relations (15) and (16) and for the submatrices (20) for a planar triangular element.

By taking a variation of the functional (18) and applying the theorem a system of two matrix equations is obtained:

$$[C_1]\{\theta_1(t)\}+[K_1]*\{\theta_1(t)\}=[C_1]\{\theta_1^0\}+[L_{12}]*\{\theta_2(t)\}+\{Q_1(t)\}, \tag{21a}$$

$$[C_2]\{\theta_2(t)\}+[K_2]*\{\theta_2(t)\}=[C_2]\{\theta_2^0\}+[L_{21}]*\{\theta_1(t)\}, \tag{21b}$$

whose solution is the temperature field for the entire system of finite elements as a function of t.

To be able to organize a step-by-step procedure for solving Eqs. (21) in time it is assumed that at the nodes the change of temperature on the time interval $[t_k, t_{k+1}]$ is approximated by

$$\{\theta(\tau)\}=A_0+\tau A_1 \quad (t_k\leqslant\tau\leqslant t_{k+1}), \tag{22}$$

where A_0 and A_1 are some constants.

If one now determines $\{\theta(\tau)\}$ in terms of the finite points of this interval one obtains

$$\{\theta(\tau)\}=\frac{t_{k+1}-\tau}{\Delta t}\{\theta(t_k)\}-\frac{\tau-t_k}{\Delta t}\{\theta(t_{k+1})\}\ (\Delta t=t_{k+1}-t_k). \tag{23}$$

Taking into account (23), the convolution in Eqs. (21) can now be computed. Indeed, by integrating over $[t_k, t_{k+1}]$ one obtains

$$[K]*\{\theta(\tau)\}=[K]\left(\frac{1}{2}\Delta t\{\theta(t_{k+1})\}+\frac{1}{2}\Delta t\{\theta(t_k)\}\right), \tag{24a}$$

$$[L]*\{\theta(\tau)\}=[L]\left(\frac{1}{2}\Delta t\{\theta(t_{k+1})\}+\frac{1}{2}\Delta t\{\theta(t_k)\}\right). \tag{24b}$$

The thermal-force can be dealt with as in (20d); however, if the heating capacity of the sources is given then the vector can easily be integrated with respect to t.

Then from (21) and using (24), and in view of the last remark one finally obtains the equations for the temperature field of a system of two bodies at the instant t_{k+1}:

$$\left([C_1]+\frac{\Delta t}{2}[K_1]\right)\{\theta_1(t_{k+1})\}-\frac{\Delta t}{2}[L_{12}]\{\theta_2(t_{k+1})\}=\left([C_1]-\frac{\Delta t}{2}[K_1]\right)\{\theta_1(t_k)\}+\frac{\Delta t}{2}[L_{12}]\{\theta_2(t_k)\}+\{Q_1\}\Delta t, \tag{25a}$$

$$\left([C_2]+\frac{\Delta t}{2}[K_2]\right)\{\theta_2(t_{k+1})\}-\frac{\Delta t}{2}[L_{21}]\{\theta_1(t_{k+1})\}=\left([C_2]-\frac{\Delta t}{2}[K_2]\right)\{\theta_2(t_k)\}+\frac{\Delta t}{2}[L_{21}]\{\theta_1(t_k)\}. \tag{25b}$$

NOTATION

$\theta(\mathbf{x}, t)$	is the temperature;
$\theta_{,i}$	is the partial derivative with respect to spatial coordinate x_i ($i = 1, 2, 3$);
$\theta_{,ii}$	is the Laplacian;
$\dot{\theta}$	is the partial derivative with respect to time;
c	is the heat capacity;
γ	is the density;
λ	is the heat-conduction coefficient;
$w(\mathbf{x}, t)$	is the specific power (per unit mass) of internal heat sources due to plastic deformation;
n_i	is the direction of the unit outer normal to the boundary;
α	is the heat-transfer coefficient.

LITERATURE CITED

1. O. C. Zienkiewicz and Y. K. Cheung, The Finite Element Method in Structural and Continuum Mechanics, McGraw–Hill, London (1967).
2. M. E. Gurtin, Quart. of Appl. Math., 22, No. 3, 252-256 (1964).
3. M. E. Gurtin, Arch. for Rat. Mech. and Anal., 16, No. 1, 34-50 (1964).
4. E. L. Wilson and R. E. Nickell, Nucl. Eng. and Des., 4, No. 3, 276-286 (1966).
5. V. A. Ditkin and A. P. Prudnikov, Integral Transformations and Operational Calculus [in Russian], Fizmatgiz, Moscow (1961).